Composite Material Mechanics: Structural Mechanics

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Introduction

THE purpose of this Survey is to review and bring together in an orderly fashion some of the principal contributions to the field of structural mechanics of structures containing composite materials. The topics of micromechanics and fracture, while quite important, are not considered in this Survey. Emphasis is given to the macromechanical structural analysis of various structural elements, including response under conditions of stable static loading, buckling, and dynamics.

The Survey unfolds in the following sequence: Straight and Curved Laminated Bars, Laminated Plates, Laminated Shells, Sandwich Structures, Applications to Practical Structural Systems, and Future Trends.

The authors hope that this contribution will be a useful reference tool for researchers and engineers already involved in structural aspects of advanced composites, as well as for those who are just entering the field.

No Survey can do full justice to such a wide field as composite-material structural mechanics. The references cited give only a glimpse of the extensive literature in this field. The authors apologize for not citing a number of important contributions in the field.

Straight and Curved Laminated Bars

Straight and curved bars are among the most widely used structural elements. However, somewhat surprisingly, the theory of laminated bars is the least developed aspect of composite-material structural theory.

Here a *bar* is defined to be a member which is relatively long along one axis, which may be either straight or curved, and relatively compact in cross section in planes perpendicular to the axis. If a bar undergoes bending deformation, due to distributed normal loads or concentrated shear forces or bending moments, it is called a *beam*. A bar subjected to axial compression load only is called a *column* (of course it undergoes bending deformation during buckling). If a bar is subjected to a combination of bending loads and axial compression, it is known as a *beam-column*; when subjected to combined bending loads and axial tension, it is called a *tie bar*.

A wide variety of approaches to the problem of laminate bending have been used; in order of increasing accuracy and sophistication, they are: 1) mechanics of materials, 2) nonhomogeneous elasticity theory, and 3) microstructural theory.

The mechanics-of-materials approach was pioneered by Hoff.[1] A more general presentation has been introduced by Berkowitz.[2]

Clark analyzed beams[3] and columns[4] which were partially delaminated in that they were joined only at a finite number of points along the length, instead of being continuously bonded.

All of the existing analyses of laminated beams by two-dimensional nonhomogeneous elasticity theory suffer from various simplifying assumptions. Cheng[5] and Gerstner[6] considered only end shear loading, while Lekhnitskii[7] and Schile[8] also considered loading by an end moment. The analysis of Cheng[5] neglected thickness-normal deformation and Schile[8] and Gerstner[6] assumed that each layer is isotropic in the plane of bending. Thus, it appears that the only existing analysis applicable to beams laminated of filamentary-composite-material layers is that of Lekhnitskii,[7] who assumed that each layer is specially orthotropic in the plane of bending. Cheng's analysis is limited to symmetrically laminated beams. The work of Schile and Gerstner was claimed to be arbitrary but actually was applied only to the case of three layers symmetrically laminated. Again, Lekhnitskii's work is more general and his specific numerical example was unsymmetrically laminated (two layers).

Pagano[9,10] presented exact analyses of laminates in cylindrical bending, i.e., bending into a cylindrical surface.

Bending of nonhomogeneous bars has been treated as a problem in flexure in the three-dimensional theory of elasticity. For the case of a constant Poisson's ratio (i.e., all layers having the same Poisson's ratio), reference is made to the work of Muskhelishvili[11] and Schile.[12] The technologically more im-

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portant case of different Poisson's ratios in different layers is much more complicated.[13,14]

The microstructural approach was first applied to laminated composite structures by Bolotin,[15] who considered plates. However, it is easy to reduce his work to the simpler case of a beam, as was done recently by Achenbach and Zerbe.[16]

The in-plane bending problem is one in two-dimensional anisotropic elasticity. It has been solved by Hashin[17] using the Niedenfuhr-Neou polynomial stress function method.

Perhaps the earliest treatment of the torsion problem for a straight bar with a nonhomogeneous cross section is due to Muskhelishvili,[18] who used the complex-variable approach. Unfortunately, this method is limited to cases for which transformations are known to map the cross section conformally onto the unit circle. Ely and Zienkiewicz and Plunkett derived a generalization of the Prandtl membrane analogy to handle nonhomogeneous cross sections,[19,20] Ely and Zienkiewicz[19] solved the problem by the finite-difference relaxation method, while Sparrow and Yu[21] used the least-squares boundary-collocation technique.

The authors are aware of only a few torsional analyses of laminated rectangular-section bars.[14,18,19] All of these treat only the case of a two-ply laminate. Numerical results were given in Refs. 18 and 19 only; they differed by less than 2% for shear-moduli ratios, $G^{(2)}/G^{(1)}$, of 1 to 3. In these analyses, each of the two constituent materials is assumed to be isotropic. These analyses are applicable to bars having layers made of filamentary composites with the filaments oriented parallel to the axis of the bar, since such a composite is transversally isotropic with the cross-sectional plane as the plane of isotropic symmetry. However, if the filament makes an angle with the axis of the bar, the material is specially orthotropic in the cross-sectional plane. Apparently the only nonhomogeneous torsional analysis of a bar having an orthotropic layer is due to Payne,[22] who considered the case of a bar composed of two rectangular regions, one isotropic and the other specially orthotropic.

Curved bars and rings are used extensively as structural elements, such as bents, roof arches, stiffening rings, etc. In composite-materials technology, they are also widely used as test specimens, usually in the form of a short ring cut from the end of a filament-wound circular tube.[23,24] Unfortunately there is a great lack of ring and curved bar analyses appropriate to those made of composite materials.

Plane elasticity analyses of in-plane bending have been carried out for isotropic-material curved bars (see many elasticity theory textbooks) and for cylindrically orthotropic curved bars.[7] So far as is known, the only ring-theory analysis which specifically considers arbitrarily laminated cross sections is the buckling analysis carried out by Keim.[25]

Laminated Plates

There are many kinds of theories and levels of sophistication in the theory of plates. First, there are linear and nonlinear theories. Within each of these theories, there are three distinct levels: 1) Thin plate analysis (analogous to mechanics-of-materials analysis in laminated bars); 2) Moderately thick plate analysis (including thickness-shear deformation, and sometimes thickness-normal deformation); and 3) Classical elasticity (nonhomogeneous in the case of a laminated plate).

The thin plate analysis has been the most popular, probably due to its simplicity. However, even this type of analysis has two subclasses: a) simplified [cf., Refs. 1 and 26] correctly applicable only to a plate laminated symmetrically with respect to its middle plane, and b) complete analysis [cf., Refs. 27 and 28] including laminate bending-stretching coupling and thus applicable to arbitrarily laminated plates, i.e., either symmetrically or unsymmetrically laminated ones.

The laminate bending-stretching coupling effect is a coupling between in-plane stretching (or membrane action) and plate bending. This effect was recognized in Russia as early as 1953 by Ambartsumyan,[29] who was concerned with unsymmetrically laminated orthotropic shells. It was also recognized independently in the U.S. by Reissner[30] at about the same time. This laminate coupling effect is the major difference between the macroscopic structural behavior of an arbitrarily laminated plate (either isotropic, specially orthotropic, or anisotropic) and homogeneous plates (either isotropic, specially orthotropic, or anisotropic), such as treated in books by Timoshenko and Woinowsky-Krieger[31] and Lekhnitskii.[7]

Werren and Norris[32] have shown that it is possible to orient the layers of multiple-layer laminates in such a way that the resulting elastic stretching behavior is isotropic, i.e., has elastic coefficients which are independent of orientation in the plane. The conditions which must be met by the laminating scheme to achieve this result are as follows: 1) the total number of layers (n) must be three or more; 2) the individual layers (denoted by index k, ranging from one to n) must have identical orthotropic elastic coefficients and thicknesses; 3) a typical layer k must be oriented at an angle $\theta_k = \pi(k-1)/n$ with respect to a reference direction. Since a laminate made according to the Werren-Norris design is isotropic in regard to stretching only (submatrix $[A]$)* and not, in general, in regard to bending and stretching-bending coupling (submatrices $[D]$ and $[B]$),* this design is called *quasi-isotropic*. Further work relating to the invariant properties of anisotropic materials was carried out by Tsai and Pagano.[33]

The coupled displacement equations of motion[27,28] are reminiscent of those governing large deflections of homogeneous plates [cf., Ref. 31, p. 418], except that the operators in the latter case are nonlinear, and also those governing small deflections of shallow shells [Ref. 31, p. 559]. For plates undergoing large deflections and shallow shells, the equations are coupled by curvatures, while in laminated anisotropic plates, the coupling is due to the nature of the nonhomogeneity. Thus, in writing boundary conditions along an edge of a laminated anisotropic plate, both in-plane and transverse effects must be considered simultaneously, i.e., four boundary conditions must be prescribed for each edge.

To obtain a general solution, it is desirable to combine the two displacement equations of motion into a single high-order equation. For zero body forces, Stavsky[28] obtained an eighth-order equation in real variables.

A way to test a single-layer orthotropic material is to use two plate loading schemes: one for producing pure bending, and the other for inducing pure twisting. Two of the bending tests are necessary: one with loading at 0° to the x axis, and the other with loading at 90°. The twisting test specimen is oriented so that its edges coincide with the principal elastic directions. With the assumption of macroscopic homogeneity through the thickness direction and the Kirchhoff hypothesis, these tests are sufficient to permit determination of four independent orthotropic elastic coefficients. Such tests were conducted on plywood by Thielemann[34] and by Hearmon and Adams,[35] on laminates by Witt et al.[36]

In checks of the elastic constants for a homogeneous isotropic plate, Beckett et al.[37] noted that the plate bending and twisting tests gave discrepancies as high as 24%. To reduce this discrepancy (they achieved 8%), Beckett et al. devised a "curvature method" which uses only the relative positions of the plate in the immediate vicinity of a relatively small test region (to take advantage of Saint-Venant's principle), without recourse to assuming the mathematical form of the deflection surface.

Tsai and Springer[38] pointed out that in addition to providing a means of measuring the twisting compliance, the pure-plate-twisting test can be used to obtain additional relations among the in-plane compliances. This concept was further developed by Tsai[39] who substituted an additional twisting test on a 45°

* The elements of the stretching, stretching-bending coupling, and bending submatrices are defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij}\, dz; \quad i,j = 1, 2, 6$$

where h is the plate thickness, z is the thickness-direction coordinate, and Q_{ij} is the plane-stress reduced stiffness coefficient.

oriented plate for the second bending test mentioned previously. Advantages claimed were more efficient specimen material utilization (twisting specimens can be much smaller than bending ones) and less dependency on span-to-thickness ratios.

An alternative to static uniaxial tests for measuring the major and minor Young's moduli in tension and compression is to use the stress-wave technique used for glass fiber reinforced plastic (GFRP) studies by Abbott and Broutman.[40] This method permits both tension and compression Young's moduli to be measured on the same specimen. Their work indicated that the tension and compression moduli for unidirectionally reinforced GFRP are the same, within experimental accuracy.

The first application of Reissner-Stavsky anisotropic theory to determination of elastic coefficients was made by Tsai.[41] It is noted that although Tsai used five specimens (two in-plane, two bending, and one twisting), he obtained sufficient data to determine all of the elements of matrices $[A]$, $[B]$, $[D]$, except the in-plane shear stiffness. However, this could have been obtained either directly by means of a scissors shear test or indirectly by means of a uniaxial specimen loaded at an angle of 45° (or any angle between 30° and 60°) to a principal elastic direction. In fact, it appears that a minimum of six tests is necessary to determine all of the $[A]$, $[B]$, and $[D]$ elements in the general case. Tsai obtained experimental laminated composite parameters $[A]$, $[B]$, and $[D]$ which agreed quite well with his theoretical ones calculated on the basis of the Reissner-Stavsky theory and the single-layer elastic coefficients.

The Reissner-Stavsky theory of plate bending was first applied to problems involving other than uniformly distributed forces or couples by Stavsky[28] and by Dong et al.[42] They considered cylindrical bending of a long, rectangular plate subjected to uniform normal pressure. A more general analysis of this kind of bending was performed recently by Whitney.[43]

Broutman et al.[44] analyzed the deflection of a two-layer, anisotropic rectangular plate subjected to an arbitrary distribution of normal pressure. The flexural boundary conditions considered were simple support on all edges, while two in-plane boundary conditions were considered: free and fully restrained. Detailed calculations were carried out for a uniform normal-pressure distribution only, for which it was shown that inclusion of the bending-stretching coupling matrix had a negligible effect when the edges were unrestrained in the plane of the plate. However, for edges fully restrained in the plane, the maximum deflection was increased by approximately 60% for typical unidirectional glass-epoxy plate. In certain cases, the in-plane loading induced at the plate middle plane due to coupling exceeded the outer-fiber stress at the same planview location.

Ashton and Waddoups[45] used the energy method to formulate the problem of plate bending due to static normal pressure. Later Ashton[46] extended this work to include material-property or thickness variation with position in the plane of the plate. Also Ashton[47] presented numerical results for a variety of cases with emphasis on effects of various boundary conditions.

The problem of a simply supported rectangular plate subjected to an arbitrary normal pressure loading was solved by Whitney and Leissa[48–50] using a double Fourier series. They obtained a closed-form solution for normal pressures distributed uniformly and according to a single-term sinusoid. Numerical results, presented for cross-ply and angle-ply plates of S-glass/epoxy and graphite/epoxy, showed that bending-stretching coupling can increase the maximum deflection of a plate as much as 300%. Recently Whitney[51] presented an analysis of clamped rectangular plates subjected to uniform normal loading. Again he showed that bending-stretching coupling can significantly reduce the flexural stiffness of unsymmetrically laminated plates. Also he showed that the in-plane (membrane) boundary conditions can significantly affect the plate response for certain angle-ply lamination schemes.

It was shown by Pister[52] that a plate laminated of isotropic materials has reduced flexural stiffness given by the following expression:

$$D^{(R)} = D - (B^2/A) \tag{1}$$

Apparently, this fact, coupled with some experimental evidence, prompted Chamis (unpublished) to suggest that the following reduced flexural stiffnesses be used for laminated anisotropic plates:

$$D_{ij}^{(R)} = D_{ij} - B_{ij} A_{ij}^{-1} B_{ij} \tag{2}$$

Ashton[53] followed up the previous suggestion for the case of simply supported rectangular plates and obtained results which agreed with Whitney's closed-form results for graphite-epoxy plates within 17%. Recent work by Whitney[51] indicated that the reduced stiffness concept is even less accurate for the case of clamped edges (see also Ashton and Whitney's book[54]).

All of the analyses previously mentioned dealt with rectangular planform plates. So far as is known, the only published work on arbitrarily laminated anisotropic plates of other specific planforms is that of Kicher,[55] who considered uniformly loaded elliptical plates (circular as a special case). Finite-element analyses, applicable to any arbitrary planform of laminated plate, have been presented in Refs. 56–58 among others.

The cross-elasticity or shear-coupling effect was discussed by Stavsky.[59] Wang[60] showed that for a generally anisotropic plate (i.e., angle ply), no solutions in separable form (i.e., product of functions of x only and of y only) satisfy simple support boundary conditions exactly.

Since bending-stretching coupling reduces the effective stiffness of an unsymmetrically laminated plate subject to normal pressure loading, one would expect that it would reduce the critical buckling load. Broutman et al.[44] made the first analysis which included this effect, using the Reissner-Stavsky plate theory,[2] for a certain type of two-layer rectangular plate. They indicated that bending-stretching coupling had very little effect on buckling load.

Ashton and Waddoups[45] presented a Rayleigh-Ritz solution for buckling of arbitrarily laminated rectangular plates subject to biaxial compression and in-plane shear loads. Their analysis gave results which agreed reasonably well with experimental results for uniaxial compression[61] (clamped-clamped and clamped-simple ends) and in-plane shear[62] (clamped-clamped ends). Ashton[46] also obtained good agreement for uniaxially compressed, laminated rectangular plates with linearly taped thickness.

Chamis[63] presented a solution by the Galerkin method for buckling of rectangular plates. His results agreed reasonably well with experimental results by Kicher and Mandell[64] for uniaxially compressed plates, simply supported on the loaded edges and either simply supported or free on the sides.

Whitney and Leissa[49,50] presented a closed-form solution of the buckling of a simply supported, arbitrarily laminated, rectangular plate subjected to biaxial compression (with uniaxial compression as a special case). Their results indicated a strong dependence upon bending-stretching coupling. In a specific example, for a 2-ply ±45°-angle-ply plate subjected to uniaxial compression, the buckling load was only about 37% of that for a plate of the same thickness but with four plies.[50] Later Whitney[65] obtained a closed-form solution for unsymmetrically laminated, cross-ply plates subjected to pure shear.

Apparently the only buckling analyses of arbitrarily laminated plates of other than rectangular planform are Baumann's and Bufler's[66,67] analyses of axisymmetric buckling of circular plates with isotropic-material layers. Two additional buckling analyses of laminated media should be mentioned: Clark's analysis of plates with discontinuous bonding[68] and Biot's micro-instability analysis.[69]

As in the case of static normal-pressure loading and in buckling due to in-plane loads, the effect of bending-stretching coupling in vibration is to reduce the effective stiffness, thus lowering the natural frequencies. Apparently the first vibrational analysis including this effect was due to Pister[52] in 1959. He considered a plate arbitrarily laminated of isotropic layers.

Although Stavsky[70] formulated his coupled bending-stretching theory of laminated anisotropic plates for the vibration problem, he did not present any numerical results. Apparently the first published results of the vibrational analysis of such plates is due to Ashton and Waddoups[45] who used the Rayleigh-Ritz method to analyze rectangular plates. They compared reasonably well with experimental results for the completely free and cantilever cases. A similar analytical and experimental study, involving combinations of simply supported and free boundary conditions, was carried out by Hikami.[71]

For the case of arbitrarily laminated orthotropic plates with simple support boundary conditions, Whitney and Leissa[49,50] presented closed-form solutions for the natural frequencies. As would be expected, their results showed a strong effect of bending-stretching coupling in lowering the frequencies.

The case of clamped boundary conditions is more complicated analytically, but perhaps more representative of practical structures. Rayleigh-Ritz and experimental investigations of such structures were carried out independently in Refs. 72 and 73. Ashton[74] considered free-free edges.

Additional vibrational analyses of plates laminated of isotropic materials may be found in Refs. 75 and 76. Analyses of wave-propagation velocities in laminated plates are presented in Refs. 77 and 78. Microlaminar effects in the vibration of plates have been considered by Biot, Bolotin, and Sun et al.[79-81]

In the theory of thin plates, geometrical nonlinearity due to large normal deflections was introduced in 1910 by von Kármán,[82] who considered homogeneous isotropic plates. Pister and Dong[83] considered large deflections of plates laminated of isotropic material. They formulated the problem in terms of the normal deflection and an Airy-type stress function. This work was followed by more general analyses by Stavsky and by Whitney and Leissa.[84,49] In another paper Stavsky[85] formulated the same problem in terms of the three displacement components. In this formulation, compatibility is assured automatically.

Pao[86] obtained a closed-form solution of the theory[84] for the case of simple unidirectional bending to a constant curvature. Antielastic curling was allowed to take place, but it was shown that by appropriately laminating the plate, the antielastic curling can be minimized.

Mayberry and Bert[87] showed that a homogeneous orthotropic plate vibrational analysis including large normal deflection, was inadequate to predict the experimental change in frequency with amplitude on unsymmetrically laminated cross-ply and angle-ply plates. Thus, an analysis including bending-stretching coupling is required. Such an analysis was carried out approximately by Wu and Vinson[88] using the reduced-flexural-stiffness concept previously mentioned. The first nonlinear dynamic analysis to include bending-stretching coupling more rigorously was Bennett's analysis[89] of the dynamic stability of rectangular plates with simply supported immovable edges. Reference 90 considered nonlinear vibration of rectangular plates clamped flexurally and with arbitrary in-plane restraint.

The major higher-order effects discussed here are thickness-shear† flexibility and thickness normal stresses. It has been recognized for a long time that thickness-shear flexibility plays an important role in reducing effective flexural stiffness in laminated, filamentary composite materials.

There are two different approaches toward considering these higher-order effects: improved plate theory (such as those of Reissner[92] and Mindlin[93] for homogeneous, isotropic plates) and three-dimensional elasticity theory. Although the latter approach is the most accurate, it is quite cumbersome mathematically, especially in the case of multiple layers. Therefore, in view of its relative simplicity, major emphasis has been concentrated on the former approach.

Unfortunately, there is very little agreement on the best way to derive an improved plate theory, and theories derived by different approaches yield differing predictions of plate performance. Probably the first such theory for laminated plates was that originated by Ambartsumyan[94] in 1958 (see also Ref. 95). In this theory, the thickness-shear stresses were assumed a priori to be distributed continuously through the thickness according to a simple parabola. This may be considered to be a generalization of Reissner's plate theory[82] to nonhomogeneous plates. A similar analysis was also presented recently by Whitney.[96] Osternik and Barg[97] made separate calculations assuming a fourth-power generalized parabolic distribution.

Another approach was used by Yang, Norris, and Stavsky,[98] who assumed that the transverse shear angle is independent of the thickness coordinate, and integrated the shear stress equations of motion to obtain the governing displacement equations. Thus, their work may be considered to be an extension of Mindlin's dynamic analysis[93] from the homogeneous isotropic case to the laminated anisotropic case. Upon integration of the equations of motion, Yang et al. introduced a shape factor in an ad hoc fashion to correlate the predicted frequencies with a known three-dimensional elasticity solution. It is noted that in their theory there are three kinds of inertias, normal translational (considered in elementary thin-plate theory), rotatory (considered by Mindlin), and translational-rotatory coupling (first considered by Yang et al.). The plate may be laminated either symmetrically or unsymmetrically. Further work using this theory was reported in Refs. 99 and 100.

Stavsky[101] originated another improved plate theory which included thickness normal stresses as well as thickness shear flexibility. Unfortunately, only the symmetrically laminated, isotropic-material, static case was considered.

Another approach is to assume that the actual plate laminated of filamentary-composite-material layers can be modeled by considering a plate consisting of alternating layers of relatively rigid layers (with properties representative of the fibers) interspersed between flexible layers (with properties typical of the matrix material).[102] Such an analysis is essentially identical to multiple-core, multiple-facing sandwich plate analysis.

Of the very few three-dimensional elasticity solutions of laminated plates, Refs. 103–105 are worth mentioning. Considering a two-layer isotropic-material plate, Schile[103] used two stress functions, which allowed Young's modulus and Poisson's ratio to be arbitrary functions of the thickness coordinate. Considering each layer to be specially orthotropic, Pagano[104] solved directly the equilibrium equations expressed in terms of the displacements.

In a very recent series of papers, Srinivas et al.[106-108] treated rectangular plates simply supported by rigid knife edges on all four edges. Reference 106 treated homogeneous and multiple-ply plates laminated of isotropic materials; numerical results were given for an unsymmetrically laminated three-ply plate. In the case of homogeneous plates, their results indicated that classical thin plate theory is adequate for plates having thicknesses up to 0.05 b ($a \geqq b$) and that Reissner's plate theory[92] is good for plates having thicknesses up to 0.10 b ($a \geqq b$). However, for three-ply plates, the errors in these two theories increase with increasing ratios of the individual-layer moduli.

Reference 107 considered free vibration of homogeneous and multiple-ply plates laminated of isotropic materials. The exact solution yielded an infinite number of doubly infinite spectra of natural frequencies, compared with three doubly infinite spectra in Mindlin's theory.[93] If one is seeking only the flexural, thickness-twist, and thickness-shear frequencies corresponding to each set of mode numbers, Mindlin's theory was shown to be adequate, but the exact three-dimensional analysis is needed if the full spectrum of modes needs to be determined. For example, Mindlin's theory does not yield symmetric thickness modes.

Reference 108 considered static deflection, buckling, and free vibration of homogeneous and laminated plates of specially orthotropic materials. For reasonable values of a certain dimensionless modal parameter, the Reissner theory was found to be adequate for deflection or buckling load and the Mindlin theory was satisfactory for natural frequencies, but neither proved to be adequate for calculating the associated stress distributions.

† Following Yu's extensive usage in sandwich plates (cf., Ref. 91), the term *thickness* shear is used here rather than transverse shear, so that the term transverse can be reserved to denote the in-plane direction normal to the longitudinal direction.

Laminated Shells

In the theory of shells, there are even more theories and levels of sophistication than in the theory of plates. As in the case of plates, there are linear and nonlinear theories. Within each of these classes, there are many distinct levels; in order of increasing sophistication and accuracy, they include:

1) *Membrane theory:* Extensional effects are considered in this very simple theory, while bending rigidities are completely ignored. Hartung,[109] for example, applied this theory to composite-material shells.

2) *Very-shallow-shell theory:* Although this is the simplest theory that includes bending rigidities, it neglects the effect of tangential middle-surface displacements on the curvatures and also neglects terms of the form Q_i/R_i in the tangential equations of motion. Here Q_i is the thickness-shear stress resultant, and R_i is the radius of curvature of shell. Donnell[110] introduced this theory for cylindrical shells developing many circumferential buckling waves. It was first applied to composite-material shells by Dong et al.[111]

3) *Shallow-shell theory:* Here only the first of the Donnell assumptions is made; i.e., z/R_i is consistently neglected in comparison to unity. Here z is the thickness-direction coordinate. This theory was originated by Morley,[112] who considered cylindrical shells only.

4) *First-approximation theories:* Here the quantity z/R_i is neglected in the middle-surface strain expressions, but not in the curvature expressions. As was pointed out by Langhaar,[113] this seeming inconsistency is the shell equivalent of the Winkler-Bach curved-beam theory and is more accurate than shallow-shell theory in which z/R_i is consistently neglected. The most popular first-approximation theory is known as Love's first-approximation theory; the most rational derivation of it is due to Reissner.[114] Love's first-approximation theory has the disadvantage that it predicts the presence of certain strain components in a general shell subject to mere rigid-body rotation about an axis normal to the shell middle surface. An improved first-approximation shell theory, in which this flaw is removed, has been presented by Sanders,[115] for example.

5) *Second-approximation theories:* In these theories it is assumed that $(h/R_i)^2 \ll 1$. Variants of this theory are due to Love[116] and Flügge[117] (cylindrical only), for example. Cheng and Ho[118] were first to apply such a theory (Flügge's) to composite-material shells.

6) *Theory including thickness-normal stress and thickness-shear strain:* This theory was originated by Reissner[119] for shells of revolution and extended by Naghdi[120] to arbitrary doubly-curved shells.

7) *Three-dimensional elasticity theory:* There is not yet any general theory of this type. However, it has been developed for special cases, such as cylinders.[121]

All of the shell theories mentioned are potentially applicable to composite-material shells. However, because of the additional complexities of anisotropy and bending-stretching coupling, primary emphasis has been on the more simple theories. In fact, a special simplified version of Theory 1, called *netting analysis*, was devised for analysis of internally pressurized, filament-wound shells. In this theory, only the fibers are considered; i.e., the contribution of the matrix material to the membrane stiffness of the shell is completely neglected.[122]

Books by Ambartsumyan[29] and Librescu[123] have been devoted almost exclusively to analysis of laminated anisotropic shells. Extensive surveys on this topic have been presented by Ambartsumyan[124,125] and Habip.[126]

Cylindrical Shells

Stable static loading

Arbitrarily laminated anisotropic circular cylinders subjected to uniform internal pressure were analyzed by Datta Roy[127] and Dong et al.[111] In Ref. 111, the solution was carried through to the point of obtaining a single generalized stress function Φ for a certain class of laminations. The governing differential equation was solved for the case of axisymmetric loading, and numerical results obtained for a semi-infinite-length, angle-ply cylinder subjected to uniform internal pressure and clamped at one end. Since analyses of this type are rather lengthy, various attempts have been made to simplify them. For example, Jones and Klein[128] showed the equivalence between arbitrarily laminated isotropic-material shells (with the same Poisson's ratio in all layers) and homogeneous isotropic ones. In some work done in the past, it was suggested that a reduced shear modulus could be used in isotropic-shell equations to handle orthotropic-material shells. However, recently Parisse and Rossetos[29] have clearly demonstrated that such a procedure can lead to very erroneous results. The example they used was a two-layer orthotropic cylinder.

Since many composite-material structures, such as piping, pressure vessels, etc., are circular cylindrical in form, and since it is relatively easy to filament wind this configuration, attention has been devoted to use of circular cylindrical shell specimens to determine the elastic coefficients. For this purpose, it is desirable to load the specimen to obtain a stress state uniform over the middle surface. Such a stress state can be achieved in homogeneous circular tubes by axial loading (tension or compression), internal pressure (neglecting discontinuity stresses caused by end effects), twisting about the axis, or any combination of these.

Card[130] conducted tension, torsion, and internal pressure tests on filament-wound glass/epoxy tubes. Hom et al.[131] conducted axial tension and compression, and internal and external pressure tests. The cylinders contained both circumferentially and longitudinally wrapped layers dispersed through the thickness. Apparently the first experimental characterization of all of the stretching submatrix quantities was made by Feldman et al.[132] They conducted tests in axial tension, internal pressure, and torsion. The cylinders had two circumferential layers and one longitudinal layer.

Whitney[133] showed that the in-surface shear stiffness cannot be determined by means of a torsion tube test unless the tube is symmetrically laminated. Whitney and Halpin[134] made a complete analysis of laminated anisotropic tubes under combined loading (axial tension or compression, internal pressure, and torque). They showed that a tube, clamped rotationally at one end and free to rotate at the other end, can be used to characterize a filamentary composite material completely.

Reuter[135] studied the effect of relatively small numbers of layers on the stresses in helically wound cylinders subjected to internal pressure. Reissner and Tsai[136] studied shear-normal coupling in the pure bending, stretching, and twisting of open and closed cylindrical shells.

There has been considerable interest in two-layer circular cylindrical shells joined by a flexible adhesive. For static loadings, these have included a simple mechanical-spring model of the adhesive proposed by Spillers,[137] and Franklin and Kicher's analysis[138] of maximum bond stresses due to axial and torsional loadings. Yew and Clark[139] showed that the detailed internal interface conditions assumed in the analysis can have a significant effect on the prediction of shell performance. Yogananda[140] presented a hybrid analysis in which he used Flügge-type shell theory for the thin outer layer and a Love-type three-dimensional-elasticity stress function for the thick inner layer.

Elpat'evskii and Vasil'ev[141] considered the interply shearing action set up in the adhesive or matrix material joining any two adjacent plies in which the material-symmetry axes intersect at an acute angle. They claimed that shear strain energy is set up by the tendency for one ply to rotate counterclockwise and the other to rotate clockwise. Thus, these rotational tendencies set up self-equilibrating shear stresses in the interply region.

The three-dimensional, nonhomogeneous elasticity equations presented by Schile and Sierakowski[142] can be specialized to treat thick laminated shells. However, they applied it to smoothly varying nonhomogeneity only. Shaffer[143] considered two-layered thick-walled tubes of incompressible orthotropic material and subjected to internal pressurization.

Buckling

In spite of the limited number of analyses of laminated anisotropic cylindrical shells subjected to stable static loadings, there have been numerous buckling analyses of such shells. Apparently the first such analysis was due to Cheng and Ho,[118] who derived the buckling characteristic equation for cylinders subjected to combined axial compression, external pressure, and torsion. Although their analysis is rather lengthy, it is based on a very accurate Flügge-type shell theory. Their general analysis was later applied to obtain numerical results for buckling under these kinds of loadings: external radial pressure,[144] combined external radial pressure and torsion,[144] axial compression,[145] combined axial compression and external radial pressure,[145] torsion,[146] combined torsion and axial compression,[146] bending,[146] combined bending and axial compression,[146] combined bending, axial compression, external radial pressure, and torsion.[147]

Reference 116 described tests conducted on 45 glass/epoxy filament-wound cylinders subjected to various loadings. The experimentally determined values ranged from 67% to 90% of predicted values in all cases except pure torsion, in which case the experimental values were generally *higher* than predicted. Card and Peterson[148] and Card[149] reported on the results of 51 tests. The experimental buckling loads averaged approximately 85% of analytical predictions based on a Donnell-type orthotropic analysis. Tasi et al.[150] obtained experimental buckling loads ranging from 65% to 85% of Donnell-type analytical predictions.

Other buckling analyses of laminated anisotropic cylinders under axial compression include Love first-approximation theory analyses of two-layer cylinders by Boresi et al.[151] and Taylor and Nickell.[152] Stavsky and Friedland[153] used an accurate Reissner-type theory for axisymmetric buckling and Love first-approximation theory for the unsymmetric modes. In another paper[154] these same investigators showed the effects of a free edge in drastically reducing the axisymmetric buckling load. Holston[155] made calculations for finite-length, axially loaded boron-epoxy cylinders with three different lamination schemes: "quasi-isotropic," "quasi-orthotropic," and "anisotropic."

Jones[156] presented exact solutions for simply supported shells under combined axial compression and external pressure. Later he considered two-layered shells with a circumferentially cracked, unbonded layer.[157] Using Flügge-type shell theory, Holston[158] analyzed buckling of laminated cylinders containing a soft elastic core (typical of a solid-rocket propellant grain) and subjected to external radial pressure, axial compression, and torsion.

All of the analyses mentioned here are for loadings which also produce buckling in homogeneous, isotropic cylinders (although at different values of loading, of course). However, Pagano et al.[159] showed both theoretically and experimentally that buckling can occur in anisotropic (off-axis wrap) cylinders subjected to axial tension. This seemingly strange phenomenon is really a shear-type buckle due to the shear deformation induced by shear coupling.

All of the analyses mentioned so far have considered perfect shells undergoing small deflections. A buckling analysis including geometric imperfections and large deflections was made by Khot.[160,161] His results indicated that any composite-material shell is less imperfection sensitive than an isotropic one (this agrees with the 67%–90% experimental/analytical values previously mentioned, as compared with about 30%–80% for isotropic shells).

Dynamics

Bert and Egle[162] and Leissa[163] presented extensive surveys on the dynamics of certain shell-type structures, including laminated shells. Apparently White[164] was first to consider the dynamics of laminated cylindrical shells. However, he did not really account for bending-stretching coupling, since he used the concept of an equivalent single-layer specially orthotropic shell. Weingarten[165] investigated experimentally the vibrations of symmetrically laminated (three-layer) isotropic-material shells with ends embedded in a flexible potting compound.

Dong[166] presented a general Donnell-type shell-theory solution for the natural frequencies of arbitrarily laminated specially orthotropic cylindrical shells with various boundary conditions. The nodal pattern used was the checkerboard pattern, which is characteristic of isotropic shells. Bert et al.[167] conducted a similar analysis using the more accurate Love first-approximation shell theory. Their results generally agreed quite closely with those of Dong, except that their lowest frequencies were lower than his.

The vibrational analysis presented by Kunukkasseril[168] was the first one to consider shells having general anisotropy (i.e., orthotropic material at arbitrary orientation). He showed that the usual checkerboard nodal pattern cannot be a solution of the governing differential equations for such shells due to the presence of the cross-elasticity terms. Using a helical-mode solution, he studied wave propagation associated with three kinds of modes having these predominant motions: radial, axial, and torsional. For finite-length shells, only two special kinds of modes were considered: axisymmetric (exact analysis), and inextensional (approximate analysis by the Rayleigh method). Apparently Ref. 167 was the first analysis of unsymmetric modes of an arbitrarily laminated, generally anisotropic shell of finite length. The idea was to combine two helical modes in such a way that a supported boundary condition (zero normal deflection) is met at both ends.

Numerous investigators have analyzed steady-state harmonic wave propagation in layered cylindrical shells [cf., Ref 169] without shear coupling; however, an analysis by Reuter[170] included this effect.

Relatively few analyses are available for analyzing the forced vibrational response of multilayer cylindrical shells. Bushnell[171] considered multiple isotropic layers and Keeffe and Windholz[172] treated multiple specially orthotropic layers.

Dynamic analyses of thick-walled, two-layered circular tubes have been performed; however, they have been limited to steady-state harmonic wave propagation in infinitely long shells. Whittier and Jones[173] considered axisymmetric modes only. For isotropic materials Refs. 174–176 treated symmetric and unsymmetric modes. For orthotropic materials, Ahmed[177] analyzed axisymmetric modes, while Subramanian[178] considered torsional modes also. The dynamic bond-stress problem has received a fair amount of attention, as evidenced by the work of Refs. 179–181, for example.

Noncylindrical Shells

Stable static loadings

Dong[182] presented a perturbation approach and applied it to shells of revolution.[183,184] A more elegant method of solution was presented by Vorovich.[185] Spirally orthotropic shells of revolution were analyzed by Kingsbury and Brull[186] and Reissner and Wan.[187]

Various special meridional geometric configurations have been analyzed by different investigators. For example, Ivanov[188] treated spherical shells, ellipsoids of revolution, and optimal-design uniform-thickness end closures for cylinders. Bessaratov and Rudis[189] considered toroidal shells.

The finite-element method of numerical analysis has been gaining considerable popularity, especially for handling practical, nonclassical shell configurations of industrial importance with odd-shaped cutouts, localized stiffeners, etc. Apparently the first application of this technique to laminated shells was made by Dong[190] in 1966, using conical-frustum shell elements.

Buckling

In contrast with cylindrical shells, very few buckling analyses of laminated anisotropic shells of noncylindrical configuration have been made. In an early analysis, Grigolyuk[191] considered two-layer isotropic conical shells subjected to uniform normal pressure. Seide[192] and Almroth et al.[193] presented numerical analyses for buckling in shells of revolution with various wall constructions, including anisotropic. Cohen[194] analyzed the effect of axisymmetric initial imperfections on the buckling of such shells.

Various investigators treated buckling of doubly curved shells of specific geometric configuration. Burmistrov and Mel'nichenko[195] considered spherical shells, Priddy[196] ogival shells of revolution, and Langhaar et al.[197] hyperboloidal shells of revolution.

Apparently the first works to include the effect of unsymmetric lamination (bending-stretching coupling) on buckling of doubly curved shells are the recent analyses of Refs. 198 and 199. They considered buckling under axisymmetric loadings of the following types of shells, using shallow shell theory: barrels, cylinders, inverse barrels (hyperboloids), and spheres.

Dynamics

Research on the dynamics of multilayer shells of geometrical configurations other than cylindrical has also been limited. Kalnins[200] described what modifications must be made to his previous free and forced vibrational analyses[201] of homogeneous, isotropic shells of revolution to apply them to symmetrically laminated, specially orthotropic shells.

Klein[202] presented a finite-element stiffness-matrix analysis for multilayer isotropic shells of revolution subjected to impulsive loading. Dong and Selna[203] applied the finite-element method to free vibrations of laminated, specially orthotropic shells of revolution. Thompson and Bert[204] presented a shallow-shell finite element applicable to arbitrarily laminated shells of arbitrary geometry. They applied it to free vibrations of doubly curved shells of revolution, as well as to noncircular cylinders and noncircular conical segments.

Most of the analyses are applicable to specific kinds of vibrational modes of specific shell configurations. Hoppmann and Baker[205] and Naghieh and Hayek[206] treated extensional (membrane) vibrations of a complete spherical shell, while Hoppmann[207] considered the same kind of modes in a spherical shell containing a small circular hole. Hoppmann and Miller[208] analyzed the flexural vibrations of a shallow spherical cap, but gave numerical results for the case of isotropic material only. Naghieh and Hayek[209] treated extensional vibrations of a spherical shell filled with a liquid. Penzes[210] made a Galerkin-method analysis of the extensional vibrations of oblate spheroidal shells.

Apparently the first vibrational analyses of unsymmetrically laminated doubly curved shells are the shallow-shell-theory analyses presented in Ref. 198, 211. They considered barrel, cylindrical, hyperboloidal, and spherical shells, including the effects of preload. Certain errors in the work of Ref. 198 were mentioned in Ref. 211.

Apparently the first large-deflection analysis of general laminated anisotropic shells was Stavsky's 1963 very-shallow-shell analysis.[212] He reduced this theory to a set of coupled nonlinear PDE's in the normal deflection w and Airy-type stress function F, but did not present any solutions. More recently, Stavsky[213] extended Reissner's large-deflection axisymmetric theory[214] of homogeneous, isotropic shells of revolution to laminated anisotropic ones.

Sandwich Structures

Sandwich structures are straight or curved beams, plates, and shells having thin, stiff facings and thick, shear-flexible cores. There are three kinds of sandwich configurations: 1) Ordinary sandwich construction, which consists of two facings surrounding a single core; 2) Open-face sandwich construction (sometimes called half-sandwich), which consists of only one facing and one core; 3) Multicore sandwich construction, which consists of multiple cores and multiple facings.

The purpose of sandwich construction is to achieve a stiff, light-weight structure. This is accomplished by using thin, high-strength, high-modulus material for the facings and low-density material for the core. The function of the core is to separate the facings so that they are at a sufficient distance from the middle surface to carry appreciable load. Thus, the analogy between an ordinary beam and an I-beam is appropriate. However, just as in the case of an I-beam, there is a small price to pay in order to achieve the advantages of separating the facings or flanges: this is the thickness-shear flexibility of the core (or web).

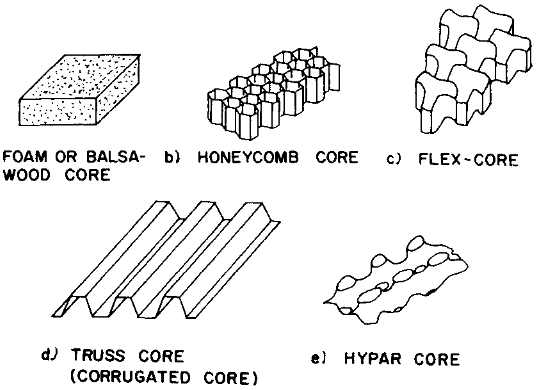

Fig. 1 Some configurations of sandwich cores.

For an ordinary sandwich with identical facings, the sandwich is symmetric about the middle surface and thus no bending-stretching coupling takes place. However, in any open-face sandwich, an ordinary sandwich with different facings, and in an unsymmetrically laminated multicore sandwich, bending-stretching coupling occurs.

A great variety of design configurations have been used for sandwich cores. Some of these are shown schematically in Fig. 1. In the original production aircraft application of sandwich construction, the British World War II de Havilland "Mosquito" bomber, a balsa-wood core was used in conjunction with plywood facings. Sometimes polyurethane foam is used as a core because of either its shock-mitigating or its thermal-insulating characteristics. Hexagonal-cell honeycomb core is presently the most popular core configuration, having been used in structural panels for numerous types of aircraft, production helicopter rotor blades, and the Apollo spacecraft. The flex core design is an attempt to achieve a core which has the same properties in both directions in the plane. Truss core is familiar to everyone in the form of corrugated cardboard boxes. The hypar (short for hyperbolic paraboloid) core design is a relatively new concept.[215] The material of which the core is made may be a polymer, aluminum, titanium, steel, or a composite material.

Materials used for sandwich facings include: glass-fiber-reinforced polymers (for housing, for light aircraft, radomes, etc.), fiberboard (for housing panels), aluminum (widely used for aircraft sandwich), titanium (for higher performance aircraft applications), and stainless steel (for high-speed aircraft panels).

Two books[216,217] and an extensive military handbook[218] are devoted to structural analysis and design of sandwich-type structures. Also chapters by Hackman[219] and by Schwartz and Rosato[220] and survey papers by Habip[221,222] should be mentioned. Here emphasis is placed on certain aspects not emphasized sufficiently or even covered at all in them, namely: 1) refined shear theory, 2) multicore configurations, and 3) composite-material facings. The latter behave like either specially or generally orthotropic materials. Inclusion of specially orthotropic cores (necessary to model honeycomb) is much simpler than inclusion of specially orthotropic facings, since the core is assumed to be capable of carrying thickness-shear stresses only.

Four modes of instability failure are possible in sandwich structures. General buckling is analogous to Euler buckling of a column; shear crimping is actually a form of general buckling which occurs in the shear mode. Face wrinkling is a local or short-wavelength buckling phenomenon. Face dimpling is actually a general buckling phenomenon involving the facing material covering a single honeycomb cell.

Multicore sandwich beams were analyzed by Kao and Ross[223] who considered two-core beams subjected to static loading. They obtained a solution for the case of simple supports. Roske and Bert[224] analyzed the free vibration of two-core beams and achieved good agreement with experimental frequencies and modal shapes.

Classical sandwich column theory is an application of Engesser's shear-flexible column theory (see Timoshenko and Gere[225]). However, Haringx[226] showed that there is an error in the Engesser theory, which he corrected in his own theory. Apparently the first application of this refined shear theory to sandwich structures was by Harris and Nordby.[227]

A bending theory of multicore plates was presented by Liaw and Little,[228] and Wong and Salama[229] applied it to buckling. The theory of Liaw and Little was extended to the case of anisotropic facings by Azar.[230]

Small static deflections of sandwich plates with anisotropic facings (usually specially orthotropic) were analyzed by Ueng and Lin,[231] Ueng,[232] and Librescu,[233] while Ueng and Lin[234] also presented a nonlinear analysis. Vibrations of such plates were analyzed in Refs. 235–237. Abdulhadi[238] treated free vibrations of multicore orthotropic sandwich plates.

Buckling of sandwich plates with orthotropic facings has been considered in Refs. 239–243 for the case of uniaxial compression loading, in Refs. 243 and 244 for in-plane shear loading, in Ref. 216 for biaxial compression, and in Ref. 245 for combined shear and compression. In Ref. 246 the refined Haringx thickness-shear-flexible theory was applied to uniaxially compressed sandwich plates with orthotropic facings.

The basic theory of sandwich shells of arbitrary geometrical configuration is generally attributed to Reissner[247] in 1949. For circular cylindrical shells, Stein and Mayers[248] extended Reissner's theory to include orthotropic facings as well as core, using Donnell-type shallow shell assumptions. This single-core sandwich shell theory was extended to multiple-core sandwich shells by Kao[249] who treated circular cylindrical shells with isotropic facings, Azar[250] who considered circular cylindrical shells with specially orthotropic facings, and Liaw[251] who considered conical shells with specially orthotropic facings.

Another extension of the basic Reissner theory was made by Martin[252] who considered small deflections of arbitrary shells with laminated generally anisotropic facings. One of the very few published analyses of sandwich shells undergoing large deflections was Schmit and Monforton's finite-element analysis[253] of circular cylindrical shells with laminated facings.

Reference 162 presented an extensive discussion (including 46 references) of the dynamics of sandwich shells of all types. Thus, here only the readily accessible references published since Fall 1969 are discussed. Patel and Hayek[254] analyzed free vibrations of freely-supported cylindrical shells of cellular sandwich construction, while Padovan and Koplik[255] treated both open and closed cylindrical sandwich shells. For sandwich cylinders, Schoenster[256] compared theoretical natural frequency predictions with those measured by the Kennedy-Pancu technique. Smirnov[257] treated flutter of sandwich cylinders with various boundary conditions. Johnson and Bauld[258] treated the dynamic stability of sandwich cylindrical panels.

Free vibrations of sandwich conical shells with orthotropic facings were treated in Refs. 259 and 260, using the Rayleigh-Ritz and Galerkin techniques, respectively. Culkowski and Reismann[261] analyzed a sandwich spherical shell undergoing axisymmetric static and dynamic loading.

Regarding buckling of axially compressed circular cylindrical sandwich shells with orthotropic facings, it appears that the first analysis was the nonlinear one of March and Kuenzi.[262] However, it was later shown in Ref. 263 that the modal shape assumed in Ref. 262 is physically invalid, and a physically valid linear analysis, based on the sandwich-shell theory of Stein and Mayers[248] was presented. In the derivation in Ref. 263, it was assumed that the Poisson's ratios associated with bending and extension are equal. This resulted in a slight algebraic simplification; however, in practice, the effect of this simplifying assumption is not restrictive. Reference 264 presented an analysis in which this assumption is not made.

In Ref. 265, an extensive numerical investigation was made, using the analysis of Ref. 263. Thre distinct modes of buckling were found: 1) axisymmetric facing buckling mode, 2) axisymmetric core shear buckling mode, and 3) unsymmetric facing buckling mode. Expressions for the dimensionless buckling coefficient were given for each kind of buckling mode.

March and Kuenzi[266] presented a linear analysis of buckling of orthotropic-facing circular cylindrical shells subjected to torsion. For the same kind of shells, Reese[267] formulated the buckling problem for axial compression, bending, torsion, or any combination of them. However, he presented numerical results for only axial compression, bending, and the combination of these two loadings, and for simply-supported and clamped edges.

Surprisingly little attention has been devoted to analysis of buckling of cylindrically curved sandwich panels with orthotropic facings: Pope[268] considered this case for axial compression loading and Refs. 244 and 245 covered in-surface shear loading and combined shear and compression.

Buckling of sandwich conical shells with orthotropic facings was treated recently by Reese[267] who formulated the problem for axial compression, bending, torsion, and any combination of these loadings. However, he presented numerical results only for pure axial compression and for pure bending. So far as the authors are aware, no other types of sandwich shells with orthotropic facings have been analyzed for buckling.

Table 1 Buckling experiments on sandwich shells with orthotropic facings

Ref.	Configuration	Loadings
269	Cylindrically curved panel	Axial compression
	Circular cylinder	Axial compression; bending; torsion; transverse shear; combined bending, and transverse shear
270	Cylindrically curved panel	Axial compression; in-surface shear
263, 270	Circular cylinder	Bending; torsion; combined bending and torsion
263	Truncated circular cone	Bending; torsion; combined bending and torsion

Apparently the only buckling experiments reported in the literature for sandwich shells with orthotropic facings are those of Refs. 263, 269, and 270. In all cases, the shells had glass-fabric reinforced plastic facings and hexagonal-cell honeycomb cores. The various geometric configurations and loadings tested are listed in Table 1. Reference 269 obtained a critical buckling stress in bending 1.99 times that in axial compression. A parabolic interaction relation was established in Ref. 269 for combined bending and transverse shear and in Ref. 263 for combined bending and torsional shear.

Applications to Practical Structural Systems

Low-density, high-modulus composites are being viewed with an eye toward eventual use in aerospace, propulsion, marine, and ground transportation systems. Up until recently, integration of advanced composites with such systems has been unfulfilled, except within the prototype hardware concept. It now appears, however, that we are rapidly approaching an era where the large-scale application of composites will soon become both technically feasible and economically attractive.[271] Table 2 summarizes the principal over-all relative advantages between advanced composites and conventional metallic structures.

Table 2 Relative advantages of advanced composites vis-a-vis conventional metallic structures

Advantages	Disadvantages
Weight, strength efficiencies	Complications in the design process
Design versatility in orienting properties	Reliability and control of design allowables
Fabrication of large and complex structural components	Complicated manufacturing equipment
	Current level of material costs

There are three basic approaches to the utilization of advanced fibrous composites in structural components. The first of these is the "direct substitution" technique and has thus far been used almost entirely in the aerospace industry. In this approach an existing metallic component is redesigned using composites, but the load, geometric, and environmental limitations imposed on the original metallic component are carried over to the composite component without change. This approach usually leads to a modest improvement in over-all structural efficiency.[272]

The second approach which was pioneered by NASA Langley Research Center[273,274] is called "selective reinforcement." In this approach, one adds a small amount of high-modulus, low-density composite (such as boron/epoxy, boron/aluminum, or graphite/epoxy) at key locations where the increased stiffness can be used most effectively without having to redesign the whole structure. A typical example of design using this approach is to add unidirectional composite material to the outstanding flanges of the stiffeners in a typical stiffened structural panel. It has been demonstrated that this approach is very effective in terms of increasing buckling loads and increasing the flutter speed.[275,276] Other advantages are increased fatigue life, minimum use of the more expensive composite material, and ease of design and manufacture.

In the third approach, a structure or component is conceived from the outset to use composites most advantageously. Load and geometric constraints are established on the basis of an optimal design using composites. This "new component" approach has the greatest potential for bringing about rather substantial structural efficiencies.

Probably the most ambitious example of structural design using advanced composites is the F-111 aft fuselage program completed by General Dynamics-Fort Worth in late 1970.[277] Fully one-half of the 920-lb structure was fabricated of composite materials (graphite/epoxy, boron/epoxy, boron/aluminum, glass/epoxy), which resulted in an 18% weight savings compared with the counterpart aluminum and steel structure. Other structural components that have been designed for simulated environments are the Grumman F-14 horizontal stabilizer and turbine and compressor blades of metallic-matrix composites under study by Allison, General Electric, Pratt and Whitney, Solar, TRW, and others. Also various special spacecraft components and helicopter rotor blades and other fatigue-dominant components should be mentioned. These and other exploratory development programs help establish design reliability through design integration using the "direct substitution" approach. Before any substantial leap into the "new component" regime is possible, however, the factors listed as disadvantages in Table 2 must be mitigated somewhat. The design process using composites will be improved as more engineers become skilled in the use of this new class of materials, and as more flexible computational programs for orthotropic structural analysis become available. We are just now beginning to generate property data for hybrid composites (i.e., syntheses of various fibers, matrices, honeycombs, etc.), so that employment of these materials in primary structures seems somewhat distant at present. The machinery and equipment needed in large-scale manufacturing will appear as the economic need arises and as industry and government commit more capital to fabrication with advanced composites. Finally, the current level of material costs, though high, is rapidly decreasing to the point of being cost-competitive with traditional metals by the late 1970's, even for the lower cost applications.[278,279]

Future Trends

In such a fast-moving field as composite material mechanics, it is very difficult to predict significant future trends with much confidence. However, the authors believe that considerably more structural mechanics research is needed in certain problem areas and thus venture a guess that future developments may be along the following lines.

1) More attention will be paid toward predicting the strength of composites reinforced with short fibers arranged in either planar random or three-dimensional random arrays. Such composites hold great promise for low-cost, high-production products made in matched-die moulds, for example.

2) More attention will be directed toward the static and dynamic behavior of composites at large strains. In the case of a composite containing a highly elastic elastomer, these strains could be within the elastic range, or in the case of a metallic-matrix composite, they could be far beyond yield.

3) As designers become more accustomed to designing composite-material structures, it is expected that more innovative designs, including automated optimal design, will emerge to utilize better the inherent characteristics of composites. For example, crack arrestment strips, mixed composites, and similar concepts are beginning to come into use now.

4) Finally, as more and more composite-materials are used in construction of complicated shaped products, the mechanics of various composite-material production processes will be investigated much more intensively in order to achieve high production rates with minimum scrappage.

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